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QUANTIZATION WITH A GLOBAL CONSTRAINT AND IR FINITENESS OF TWO-DIMENSIONAL GOLDSTONE SYSTEMS

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A method of quantizing Goldstone systems with a global continuous symmetry by introducing a global constraint is presented. This procedure is used to study the IR finiteness of the weak coupling expansion of such models at two dimensions. In this scheme, the propagator is IR finite at any dimension and all observables are proved to be IR finite at $d = 2$. The IR properties of non-standard models are elucidated.

1. Introduction

It was recently proved that the infrared divergences which occur in the weak coupling perturbative expansion of two-dimensional field theories and statistical systems with a global continuous symmetry disappear when looking at invariant quantities [1–3], although the expansion is performed in the spontaneously broken symmetry phase, which does not exist at two dimensions [4, 5]. The proof of this fact [2], and the previous calculations on such models [6–10], involve the introduction of an extra symmetry-breaking term which makes the perturbative expansion finite; this term being set to zero at the end of calculations in order to recover the IR limit.

In this paper we adopt a different approach to this problem. We show that it is possible to introduce a global constraint in the quantization of those models in such a way that the perturbative expansion of any quantity is IR finite at two dimensions. This new quantization procedure coincides with the usual one only for invariant observables, where we recover the usual IR-finite perturbative expansion.

If this approach seems simpler in principle than the direct proof of ref. [2], we shall use, in fact, the technical results of those references to prove the IR finiteness of the new perturbative expansion. However, this new expansion leads to more complicated calculations than the usual one, so we think that its interest lies in a new insight in the IR properties of two-dimensional Goldstone systems.

This paper is organized as follows: for simplicity, we treat the euclidean O(N) non-linear σ model in detail. In sect. 2 we present the quantization with a global constraint; for sake of rigor the study is performed on a lattice regularized theory. We prove that the introduction of our constraint leads only to a change in the propagator which becomes IR finite. The IR finiteness of the corresponding perturbative

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expansion is proved in sect. 3. Finally, we discuss the case of other Goldstone models in sect. 4. In particular, our approach allows us to study the IR structure of the non-standard models introduced by Friedan in refs. [11, 12] and to prove that it is always possible to define an IR-finite perturbative limit, but not in a standard way.

2. Quantization of the O(N) σ model with a global constraint

We consider the euclidean O(N) σ model on a two-dimensional square lattice of spacing a and size L with periodic boundary conditions. The action is

$$A = \frac{1}{2t} \sum_x a^2 [\nabla_\mu \boldsymbol{S}(x)]^2 , \tag{2.1}$$

where $\boldsymbol{S}(x)$ is an N-component real field restricted on the sphere S_{N-1} by the constraint

$$|\boldsymbol{S}(x)|^2 = 1 \tag{2.2}$$

and where ∇_μ runs for the finite difference between two nearest neighbours in direction μ ($\mu = 1, 2$).

The weak coupling perturbative expansion is obtained by setting

$$\begin{aligned} S^i(x) &= \sqrt{t}\pi^i(x) \qquad i = 1, N-1 \\ &= \sigma(x) = \sqrt{1 - t\boldsymbol{\pi}^2(x)} \qquad i = N , \end{aligned} \tag{2.3}$$

and by expanding in powers of t the integral generating the partition function, which reads, up to non-perturbative terms

$$Z(t) = \int \prod_x \mathrm{d}\pi^i(x) \exp\left[-\tfrac{1}{2}a^2 \sum_x (\nabla_\mu \pi^i)^2 - \frac{a^2}{2t} \sum_x (\nabla_\mu \sigma)^2 - \sum_x \ln \sigma \right] . \tag{2.4}$$

As long as the size of the system, L, is finite, we expect no IR divergences. However, we have always $N-1$ Goldstone modes which make the propagator undefined. Indeed, the quadratic part of the action is

$$A_0 = \frac{1}{2L^2} \sum_p \hat{\pi}^i(p) \left[\sum_\mu \frac{4}{a^2} \sin^2 \frac{ap_\mu}{2} \right] \hat{\pi}^i(-p) , \tag{2.5}$$

where $\hat{\pi}(p)$ is the Fourier transform of $\pi(x)$:

$$\pi(x) = \frac{1}{L^2} \sum_p \mathrm{e}^{-ipx} \hat{\pi}(p) . \tag{2.6}$$

So A_0 has a zero eigenvalue corresponding to the $N-1$ modes $p = 0$ and is not invertible. As explained in sect. 1, this difficulty was usually removed by introducing a symmetry breaking term (this is equivalent to introduce a mass and makes the propagator well defined), then by taking the infinite volume limit and finally by using the fact that O(N) invariant quantities have an IR-finite limit when the symmetry-breaking term is set to zero [1, 2].

On the other hand, those Goldstone modes may be eliminated by introducing a global constraint which fixes the classical solution around which fluctuations are computed. The most simple way in our case is to fix the field $\boldsymbol{S}$ in a given direction at some arbitrary point x_0. So let us introduce the constraint

$$\boldsymbol{S}(x_0) = \boldsymbol{S}_0 \tag{2.7}$$

in the integral (2.4). We get

$$Z_{\boldsymbol{S}_0}(t) = \int \mathscr{D}[\boldsymbol{S}]\delta_{\boldsymbol{S}_0}[\boldsymbol{S}(x_0)] \exp -A[\boldsymbol{S}]\,. \tag{2.8}$$

$\delta_{\boldsymbol{S}_0}(\boldsymbol{S})$ is the Dirac measure at the point $\boldsymbol{S}_0$ on the sphere S_{N-1}. $\mathscr{D}[\boldsymbol{S}]$ is the integration measure $\prod_x \mathrm{d}\pi^i(x)/\sigma(x)$. In the same way, we define the average value of any function of the field $F[\boldsymbol{S}]$ with constraint (2.7) as

$$\langle F\rangle_{\boldsymbol{S}_0} = \frac{1}{Z_{\boldsymbol{S}_0}} \int \mathscr{D}[\boldsymbol{S}]\delta_{\boldsymbol{S}_0}[\boldsymbol{S}(x_0)]F[\boldsymbol{S}] \exp -A[\boldsymbol{S}]\,. \tag{2.9}$$

We now look at the corresponding perturbative expansion. Choosing $\boldsymbol{S}_0$ as the direction σ in (2.3), the constraint reads

$$\boldsymbol{\pi}(x_0) = 0\,, \tag{2.10}$$

or equivalently in dual space

$$\sum_p \hat{\pi}^i(p)\,\mathrm{e}^{-ipx_0} = 0\,. \tag{2.11}$$

If we eliminate the zero modes in (2.4) via (2.11), the quadratic part of the action is now given by (2.5), where the summation over p is restricted to $p \neq 0$ and is now invertible. Inverting A_0 and inserting constraint (2.11) in (2.6), we get for the propagator (in position space)

$$D(x, y) = \frac{1}{L^2} \sum_{p\neq 0} \frac{(\mathrm{e}^{-ipx} - \mathrm{e}^{-ipx_0})(\mathrm{e}^{ipy} - \mathrm{e}^{ipx_0})}{(4/a^2)\sum_\mu \sin^2(\frac{1}{2}ap_\mu)}\,. \tag{2.12}$$

The propagator is now well defined, but translation invariance has been broken by the constraint, so the propagator depends on x_0.

Moreover, it is easy to see that interaction terms in (2.4) are not modified by the constraint. So, in the perturbative expansion defined by constraint (2.7), the graphs are the same as in the usual perturbative expansion, only the propagator is modified and is given by (2.12).

The crucial point is that this propagator remains finite in the infinite volume limit. Indeed, as $L \to \infty$, the summation over p becomes an integration in the first Brillouin zone, but the integral is convergent at $p = 0$. So the new propagator $D(x, y)$ is really

IR finite. This procedure may, of course, be performed at any dimension $d \geq 1$ without changing this conclusion.

The previous study, performed on a lattice for sake of rigor, shows that the introduction of constraint (2.7) in the quantization leads only to a change in the propagator. This procedure may be performed whatever the ultraviolet regulator is. Indeed, from (2.12), the new propagator D is related to the usual IR-divergent one D_0 by

$$D(x, y) = D_0(x, y) - D_0(x, x_0) - D_0(x_0, y) + D_0(x_0, x_0)\,. \tag{2.13}$$

Eq. (2.13) may be generalized to any kind of regularized propagator D_0 and is sufficient to define an IR-finite propagator D (see fig. 1).

3. IR finiteness of the constrained perturbation at two dimensions

We now study the IR structure of this new perturbative expansion at two dimensions. As explained in sect. 1, for an arbitrary function $F[\boldsymbol{S}]$, its average value $\langle F\rangle_{\boldsymbol{S}_0}$ appears to be IR finite, whether F is O(N) invariant or not.

This is not a surprising fact. Indeed, the constrained average value of F (F being a function of the field $\boldsymbol{S}$ at points $x_1 \dots x_p$) may be written as the unconstrained average value of some O(N) invariant function of $\boldsymbol{S}$ at the points $x_0, x_1 \dots x_p$ (x_0 being the point where the constraint is fixed). So, the IR finiteness of the constrained perturbative expansion is equivalent to the IR finiteness of invariant observables in the usual perturbative scheme.

However, it is interesting to look at the structure of the IR divergences and at their cancellations in this new perturbative scheme: IR divergences of any graph appear to be contained only in "disconnected parts" generated by the new propagator. Up to these "disconnected parts" which give divergences in power of the volume, the remaining "connected parts" of *any* graph are IR finite.

Moreover, a simple argument connects the disconnected parts to the partition function and shows that they combine into an IR-finite contribution depending only on geometrical properties of the model.

From (2.13), any graph computed with the new propagator D may be expressed in terms of graphs computed with the usual one D_0. More precisely, let us consider some function F and some graph G of the perturbative expansion of $\langle F\rangle$; we have

$$G[D] = \sum_{G \overset{\mathscr{D}}{\to} \bar{G}} (-1)^{n(\bar{G})} \bar{G}[D_0]\,, \tag{3.1}$$

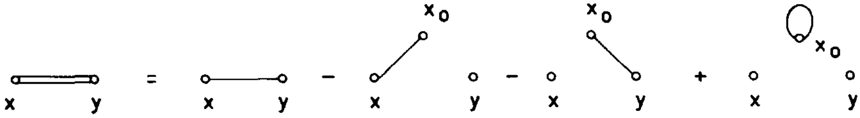

Fig. 1. The new propagator D expressed in terms of the usual one D_0.

where $G[D]$ ($G[D_0]$) is the integral of the graph G computed with the propagator D (D_0). The sum runs over all graphs $\bar{G}$ obtained from G by disconnecting an arbitrary number $n(\bar{G})$ of end points of lines of G and by attaching them at the point x_0; a factor (-1) is attached to each disconnected end point. An additional IR cut off is introduced in order to define D_0 (for instance we may take for D_0 its usual lattice form where we have eliminated the zero mode):

$$D_0(x-y) = \frac{1}{L^2} \sum_{p \neq 0} \frac{e^{-ip(x-y)}}{(4/a^2) \sum_\mu \sin^2(\frac{1}{2}ap_\mu)} . \tag{3.2}$$

It is easy to see that this operation gives zero if a derivative coupling is attached to the end point of a propagator which is disconnected. So the former operation, which shall be denoted $\mathscr{D}$, has only to be performed on end points which are not attached to derivatives of the field.

As long as the graphs $\bar{G}$ remain connected to the external vertices, the analysis of ref. [2] shows that $\bar{G}[D_0]$ would have logarithmic IR divergences*. A more serious problem comes from the graphs $\bar{G}$ which have disconnected parts (not attached to external vertices), since those disconnected parts will give contributions in power of the volume. In order to treat those divergences, we first have to give a rigorous formulation to the above considerations.

Definition

Let $\bar{G}$ be a graph obtained from G by the operation $\mathscr{D}$.

The disconnected part $V(G, \bar{G})$ associated to $\bar{G}$ in G is defined as the greatest disconnected graph $V \subset G$ which may be obtained at some stage of the operation when disconnecting the lines of G to get $\bar{G}$ in every possible order. (This defines $V(G, \bar{G})$ in an unique way).

Then, let us decompose the operation $\mathscr{D}(G \to \bar{G})$ into three steps (see fig. 2):

(a) First, we disconnect the $n(V)$ end points of lines of $(G-V)$ attached to $V(G, \bar{G})$ so that we obtain the graph V and a graph C with $n(V)$ lines attached to x_0(C may be seen as the graph obtained by shrinking V into the vertex x_0 in G). If $V = \emptyset$, $n(V) = 0$ and $C = G$.

(b) Second, we disconnect the lines of V which are disconnected in $\bar{G}$, so that we obtain a graph $\bar{V}$.

(c) Third, we disconnect the remaining lines of C. In this operation we have disconnected and attached to x_0 $n(\bar{C})$ end points of lines of C, which were not already

* When dealing with some UV regularization (in particular the lattice regularization), more serious IR divergences (in power of the IR cut off) are known to occur graph by graph and to cancel between different graphs (the measure terms being essential for such cancellations). It may be proved [13] that those divergences disappear in each graph when a first UV subtraction is performed (subtraction of quadratic divergences at zero momenta). The corresponding counterterms are in fact zero, so that the theory is not modified by those subtractions. In this section, such subtractions are assumed to be performed if necessary so that only logarithmic IR divergences are present and the analysis of ref. [2] may be performed without this additional difficulty.

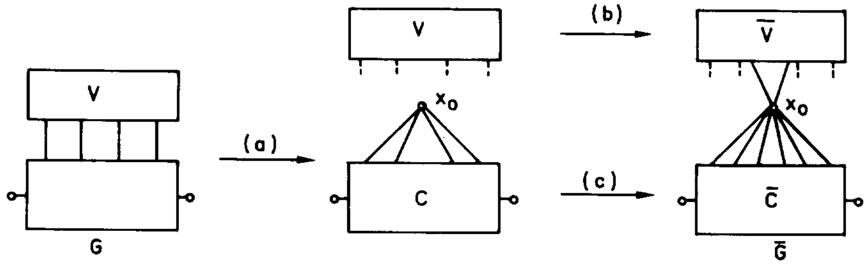

Fig. 2. The three steps of operation $\mathscr{D}$ which define the connected and disconnected parts of $\bar{G}$.

attached to x_0 in C, and without generating a new disconnected part in order to obtain a graph $\bar{C}$; let us call this operation $\mathscr{D}_C$.

Definition

$\bar{V}$ will be called the disconnected part of $\bar{G}$, $\bar{C}$ the connected part of $\bar{G}$.

Each connected component of $\bar{C}$ is attached to some external vertex; a connected component of $\bar{V}$ may only be attached to x_0.

Now, we decompose the summation over all graphs $\bar{G}$ in (3.1) as a summation over all possible $V \subset G$, times a summation over all $\bar{V}$ obtained from V by operation $\mathscr{D}$, times a summation over all $\bar{C}$ obtained from C by the operation $\mathscr{D}_C$. The summation over all $\bar{V}$ gives the integral associated to the disconnected part V, which is simply $V[D]$. Defining the integral associated to the connected part C, $C_{conn}[D]$, as

$$C_{conn}[D] = \sum_{\substack{C \xrightarrow{} \bar{C} \\ \mathscr{D}_C}} (-1)^{n(\bar{C})} \bar{C}[D_0] , \tag{3.3}$$

we finally get the decomposition into connected and disconnected parts:

$$G[D] = \sum_{(V,C)} (-1)^{n(V)} V[D] \cdot C_{conn}[D] . \tag{3.4}$$

Now we can discuss the IR structure of $G[D]$. We first consider the connected parts. We have the following result.

Lemma 1

For any connected part C in some G, $C_{conn}[D]$ is IR finite.

Proof

The graphs $\bar{C}$ have the same IR structure as the usual graphs G of the model. So we can use the results of ref. [2], which give the IR structure of any graph computed with the propagator D_0. In our notation, lemma 2.1 of ref. [2] reads:

Given any graph $\bar{C}$, the IR behaviour of $\bar{C}[D_0]$ is given by

$$\bar{C}[D_0] = \sum_{\substack{E \subset \bar{C} \\ \text{dominant}}} \text{fp}\, E[D_0] \cdot [\widetilde{\bar{C}/E}][D_0] + \text{negl. terms} . \tag{3.5}$$

In (3.5), the sum runs over all "dominant subgraphs" E in $\bar{C}$ defined by the following conditions:

(D.1): E contains all external vertices of $\bar{C}$ (including x_0).

(D.2): E has no disconnected part (each connected component of E contains at least one external vertex).

(D.3): The end points of lines of $\bar{C}$–E attached to a vertex of E do not carry a derivative coupling.

fp $E[D_0]$ is the IR-finite part of the amplitude of the dominant E, and is a finite amplitude, independent of the IR cut off.

$[\widetilde{\bar{C}/E}][D_0]$ is the amplitude of the graph obtained by shrinking E into a vertex of $\bar{C}$, and diverges logarithmically as the IR cut off is set to zero (see fig. 3).

Now, for a given C, let us consider some $\bar{C}$ and some dominant E $\neq \bar{C}$ in $\bar{C}$. We consider the set $\mathscr{E}$ of the end points of lines of $\bar{C}$ which:

(a) do not belong to E,

(b) were attached in C to vertices different from x_0, and which belong to E in $\bar{C}$.

The set $\mathscr{E}$ is not empty, otherwise $\bar{C}$–E would be a disconnected part of $\bar{C}$, which contradicts the fact that $\bar{C}$ has no disconnected part. We then consider all graphs $\bar{C}'$ obtained from $\bar{C}$ by attaching the end points belonging to $\mathscr{E}$ either to x_0 or to their original vertex in E in all possible ways. The graphs $\bar{C}'$ are always obtained from C by the operation $\mathscr{D}_C$ and are present in (3.4). Moreover, E is a dominant subgraph of any $\bar{C}'$ and the graphs $[\widetilde{\bar{C}'/E}]$ coincide. So, from (3.5) the dominant E gives the same IR divergence in each $\bar{C}'$ (see fig. 3). When summing over every $\bar{C}'$ in (3.4), it is easy to see that the divergences associated to E cancel owing to the factor $(-1)^{n(\bar{C}')}$. This argument may be applied for any possible dominant in (3.4). This ensures the result of lemma 1.

We now consider the disconnected parts V. As mentioned above, for a given V, $V[D]$ is not *a priori* IR finite. We shall prove that, summing upon different V, we get an IR finite contribution.

Given a graph G of the perturbative expansion of some function $F[\boldsymbol{\pi}]$, any possible disconnected part V in G is a graph of the observable

$$V_{\bar{n}} = \frac{1}{n!} \frac{\partial}{\partial a^{i_1}} \cdots \frac{\partial}{\partial a^{i_n}} e^{V[\boldsymbol{\pi}(x)-\boldsymbol{a}]-V[\boldsymbol{\pi}]}\Big|_{\boldsymbol{a}=0}, \tag{3.6}$$

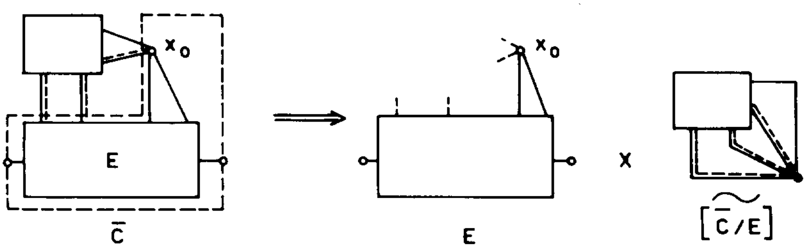

Fig. 3. The factorization of IR divergences of $\bar{C}$ in terms of essentials E as given by eq. (3.5). Lines whose end points belong to the set $\mathscr{E}$ are drawn as parallel full and dashed lines.

where $V[\pi]$ is the sum of all interaction terms in the functional integral (2.4)

$$V[\pi]=-\sum_x\left[\frac{a^2}{2t}(\nabla_\mu\sigma)^2+\ln\sigma\right], \tag{3.7}$$

and $\bar{n}$ is the set of n indices, $\bar{n}=\{i_1\ldots i_n\}$. Similarly, the corresponding graph C is a graph of the observable

$$F\Delta^{\bar{n}}(x_0)=F[\pi]\pi^{i_1}(x_0)\ldots\pi^{i_n}(x_0). \tag{3.8}$$

Counting factors and symmetry factors of the graphs manage in such a way that, when summing over all graphs G to obtain $\langle F\rangle_{S_0}$, eq. (3.4) can be generalized to

$$\langle F\rangle_{S_0}=\sum_{\bar{n}}\langle F\Delta^{\bar{n}}(x_0)\rangle_{\substack{S_0\\ \text{conn}}}\ \langle V_{\bar{n}}\rangle_{S_0}, \tag{3.9}$$

where $\langle F\Delta^{\bar{n}}(x_0)\rangle_{\substack{S_0\\ \text{conn}}}$ is the sum of the $C[D]_{\text{conn}}$ over all graphs C corresponding to the operator $F\Delta^{\bar{n}}(x_0)$, and where $\langle V_{\bar{n}}\rangle_{S_0}$ is the sum of the $V[D]$ over all graphs V corresponding to $V_{\bar{n}}$. From lemma 1, $\langle F\Delta^{\bar{n}}(x_0)_{\substack{S_0\\ \text{conn}}}$ is known to be IR finite graph by graph. We now prove that each $\langle V_{\bar{n}}\rangle_{S_0}$ is also finite.

Lemma 2

$\langle V_{\bar{n}}\rangle_{S_0}$ is simply given by

$$\langle V_{\bar{n}}\rangle_{S_0}=\frac{1}{n!}\frac{\partial^n}{\partial a^{i_1}\cdots\partial a^{i_n}}\frac{1}{\sqrt{1-ta^2}}\bigg|_{a=0}. \tag{3.10}$$

Proof

Let us consider the generating function of the $\langle V_{\bar{n}}\rangle_{S_0}$,

$$V(\boldsymbol{a})=\sum_{\bar{n}}a^{\bar{n}}\langle V_{\bar{n}}\rangle_{S_0}. \tag{3.11}$$

From (3.6), $V(\boldsymbol{a})$ is defined by

$$V(\boldsymbol{a})=\frac{1}{Z_{S_0}}\int\prod_x\mathrm{d}\pi(x)\delta[\pi(x_0)]\,\mathrm{e}^{-A_0[\pi]+V[\pi-a]}. \tag{3.12}$$

Performing the change of coordinate $\boldsymbol{\pi}\to\boldsymbol{\pi}+\boldsymbol{a}$ on the sphere S_{N-1}, we get

$$V(\boldsymbol{a})=\frac{1}{\sqrt{1-ta^2}}\frac{Z_{S(a)}}{Z_{S_0}}, \tag{3.13}$$

where $\boldsymbol{S}(a)$ is the point on the sphere defined as

$$\boldsymbol{S}(a)=\boldsymbol{S}_0(1-ta^2)+\sqrt{t}\boldsymbol{a}. \tag{3.14}$$

The factor $1/\sqrt{t-ta^2}$ comes from the constraint. Indeed, the Dirac measure on the sphere is, in our coordinate system,

$$\delta_{S(a)}(S)=\sqrt{1-ta^2}\,\delta^{N-1}(\pi-a)\,. \tag{3.15}$$

From O(N) invariance, $Z_{S(a)}=Z_{S_0}$. So $V(a)$ is obviously IR finite and (3.13) leads to lemma 2.

So, lemmas 1, 2 and (3.9) ensure the IR finiteness of any observable $F[S]$. An interesting point is that we have used the global symmetry of the model only in the proof of lemma 2, when identifying the partition functions computed with different constraints.

Finally let us discuss the relation between the results of our procedure and of the usual one. For O(N) invariant observables F, $\langle F\rangle_{S_0}$ is, in fact, independent of the constraint, and we recover the usual IR-finite result. For non-invariant observables, we have to average over all the constraints to get the physical average value of F. But it is easy to see that

$$\langle F\rangle=\int_{S_{N-1}}\mathrm{d}S_0\,\langle F\rangle_{S_0}=\langle\bar{F}\rangle_{S_0}\,, \tag{3.16}$$

where $\bar{F}$ is the projection of F upon the subspace of O(N) invariant functions defined as

$$\bar{F}[S]\int_{O(N)}\mathrm{d}R\,F[R^{-1}S]\,. \tag{3.17}$$

So, averaging a non-invariant observable over the constraint, we recover a finite invariant observable.

4. IR structure of general models

The arguments of sect. 3 may be applied without difficulties to the non-linear models which have a different global symmetry group. In this section we want to apply our approach to the IR structure of the general non-linear σ models discussed by Friedan in ref. [5]. Such models are constructed on a general (non-homogeneous) riemannian space M by the action

$$A[\phi]=\frac{1}{2t}\int \mathrm{d}^d x\;\partial_\mu\phi^i(x)g_{ij}(\phi)\,\partial_\mu\phi^j(x)\,, \tag{4.1}$$

where the field $\phi(x)$ is an element on M and where $g_{ij}(\phi)$ is the metric tensor on M at the point ϕ (in the coordinate system ϕ^i). In such general models there is no natural measure on the space of fields, so that we have to choose an *a priori* measure $d_M\phi$ on M:

$$\mathscr{D}[\phi]=\prod_x d_M\phi(x)\,. \tag{4.2}$$

So the parameters of the model are the metric g/t and the measure $d_M\phi$.

In refs. [12, 13], Friedan studied the renormalization properties of such models. He showed that those models are renormalizable at 2 dimensions and established the renormalization group equations (for the metric and the measure) at $d=2+\varepsilon$. Since there is no symmetry in these models, it is necessary to introduce a constraint to fix the point on M around which one computes fluctuations. The sorts of constraints we have presented in sect. 2 for the O(N) model (that is to fix the field at some point) have some advantages over the constraints used in ref. [12]: they are independent of the coordinate system chosen on M and it is not necessary to introduce ghost fields (the jacobian of the constraint is a constant).

So, let us consider the general model defined on M by (4.1) and (4.2). As in sect. 2, we introduce the constraint $\phi(x_0)=\phi_0$ (where ϕ_0 is some point on M), so that we define

$$Z_{\phi_0}=\int \mathscr{D}[\phi]\delta_{\phi_0}[\phi(x_0)]\,\mathrm{e}^{-A[\phi]}\,, \tag{4.3}$$

where δ_{ϕ_0} is the Dirac measure at ϕ_0, defined in a coordinate system ϕ^i as

$$\delta_{\phi_0}[\phi]=|g(\phi_0)|^{-1/2}\delta(\phi^i-\phi_0^i)\,. \tag{4.4}$$

Similarly, for any function of the fields $F[\phi]$, we define

$$\langle F\rangle_{\phi_0}=\frac{1}{Z_{\phi_0}}\int \mathscr{D}[\phi]\delta_{\phi_0}[\phi(x_0)]F[\phi]\,\mathrm{e}^{-A[\phi]}\,. \tag{4.5}$$

The conclusions of sect. 2 remain valid: the propagator is given by (2.13) and the interaction terms remain unchanged, the perturbative expansion is finite as long as the volume V is finite.

If we now take the infinite volume limit (the UV cut off being fixed), a difference with the case of models with a global symmetry appears. Indeed, the IR divergences in powers of the volume have no reason to be cancelled by the measure terms which are a free parameter of the model. (See remark in sect. 3). This is a consequence of the fact that, in general, fluctuations will (perturbatively) generate a mass of the order of the UV cut off while we expand around a massless theory. As explained in sect. 3, these volume terms are cancelled only if quadratic UV divergences are subtracted at zero momenta; power counting shows that the corresponding counterterms will appear as a renormalization of the measure $d_{\mathrm{M}}\phi$, which has to be performed in order to avoid perturbative generation of a mass. So the measure has to be adjusted order by order to keep a massless theory; the result at first order is

$$d_{\mathrm{M}}\phi=d\phi^i|g|^{1/2}\,\mathrm{e}^{tR/48+\mathrm{O}(t^2)}\,, \tag{4.6}$$

where R is the scalar curvature, which is in agreement with ref. [12].

If the measure is chosen in that way, volume terms disappear and we may apply the analysis of sect 3. We may define the operators $V_{\bar{n}}$ and $F\Delta^{\bar{n}}(x_0)$ by (3.6) and (3.8) (those operators depend on the coordinate system) and eq. (3.9) remains valid. Since lemma 1 involves only graphical arguments, the $\langle F\Delta^{\bar{n}}(x_0)\rangle_{\substack{\phi_0\\ \mathrm{conn}}}$ are IR finite, but the $\langle V_{\bar{n}}\rangle_{\phi_0}$ are IR divergent. Indeed, the proof of lemma 2 runs in the general case up to eq. (3.13), which now reads

$$\langle V_{\bar{n}}\rangle_{\phi_0}=\frac{1}{n!}\frac{\partial^n}{\partial\phi^{i_1}\,\partial\phi^{i_n}}\left[\frac{|g(\phi)|^{1/2}}{|g(\phi_0)|^{1/2}}\frac{Z_\phi}{Z_{\phi_0}}\right]_{\phi^i=\phi_0^i}, \tag{4.7}$$

but in the non-standard case there is no symmetry principle which ensures that $Z_\phi=Z_{\phi_0}$. Since powers of the volume have been eliminated by the choice of the measure, it may be proved that Z_ϕ/Z_{ϕ_0} diverges as powers of ln V. However, from (3.9) and (3.12), we deduce that

$$\langle F\rangle_{\phi_0}=\frac{1}{|g(\phi_0)|^{1/2}Z_{\phi_0}}\sum_{\bar{n}}\langle F\Delta^{\bar{n}}x_0\rangle_{\mathrm{conn}}\frac{1}{n!}\frac{\partial^{\bar{n}}}{\partial\phi^{\bar{n}}}(|g(\phi)|^{1/2}Z_\phi)_{\phi=\phi_0}. \tag{4.8}$$

So, the IR divergences of the model are entirely contained in the divergences of the partition functions Z_ϕ*.

In fact, we have yet some arbitrariness in the choice of the measure. Indeed we may add terms in $(1/V)\ln^p V$ in $d_M\phi$, so that we modify the logarithmic divergences Z_ϕ. In particular, it is possible to adjust the measure by *ad hoc* terms in $(1/V)\ln^p V$ to have

$$Z_\phi=Z_{\phi_0},\qquad \forall\phi\in\mathrm{M}. \tag{4.9}$$

From (4.8), it follows that any $\langle F\rangle_{\phi_0}$ is IR finite. The average value of F, defined as the sum over all possible constraints,

$$\langle F\rangle=\frac{\int_{\mathrm{M}}\mathrm{d}\phi_0|g(\phi_0)|^{1/2}Z_{\phi_0}\langle F\rangle_{\phi_0}}{\int_{\mathrm{M}}\mathrm{d}\phi_0|g(\phi_0)|^{1/2}Z_{\phi_0}}, \tag{4.10}$$

is also IR finite. The *ad hoc* measure to obtain (4.11) is for the square lattice model

$$d_{\mathrm{M}}(\phi)=\mathrm{d}\phi^i|g(\phi)|^{1/2}\exp\left\{tR(\phi)\left[\frac{1}{48}+\frac{1}{V}(\tfrac{1}{2}D_0(0)-\tfrac{1}{48})\right]+\mathrm{O}(t^2)\right\} \tag{4.11}$$

where $D_0(0)$ is given by (3.2) and diverges as ln V. But it is sufficient to adjust the measure in order to have Z_ϕ/Z_{ϕ_0} = an arbitrary function of ϕ and t (which is always possible) to get an IR-finite perturbative expansion for any observable. So, it is possible to adjust the measure in order to define an IR-finite limit, but this limit is not "natural", in that sense that it is not unique.

For the standard models defined on a homogeneous space M with the canonical invariant measure, eq. (4.9) is automatically satisfied, this ensures the IR finiteness, as for the O(N) model.

* Eq. (4.8) is very likely true even if the measure is not adjusted to (4.6), the $F\Delta^{\bar{n}}$ remaining finite and the divergences in power of the volume being contained in Z_ϕ.

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